Rare-event Analysis and Simulations for Gaussian and Its Related Processes

Jingchen Liu

Department of Statistics
Columbia University

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Overview

- Stochastic partial differential equations (SPDE)
- Material failure problem
- Asymptotic analysis
- Rare-event simulations

Gaussian Random Field

- Probability space $(\Omega, \mathcal{F}, P)$
- $f : T \times \Omega \to \mathbb{R}$, $f(t, \omega)$, short form: $f(t)$.
- $(t_1, ..., t_n) \subset T$, $(f(t_1), ..., f(t_n))$ is a multivariate Gaussian random vector.

Interesting quantities

- The tail probabilities of functions of $\Gamma(f(\cdot))$
- The supremum norm

$$\Gamma(f) = \sup_{t \in T} f(t)$$

- General convex functions, for instance,

$$\Gamma(f) = \int_{t \in T} e^{f(t)} dt$$

- Solutions to differential equations with coefficients driven by $f(t)$.

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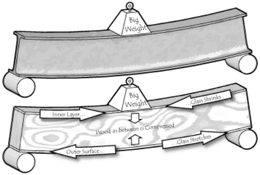
Material Failure – one dimensional example

Physical meaning

- $u(x)$: the shape of the material
- $\nabla u(x)$: strain
- $p(x)$: pressure
- $a(x)$: material-specific coefficients



[1]

[1] The picture is published at http://www.guillemot-kayaks.com

Material Failure

- The partial differential equation: $x \in T$

$$\begin{cases} -\nabla \cdot \sigma(x) = p(x) \\ \sigma(x) = a(x)\nabla u(x) \end{cases}$$

- The ordinary differential equation: $x \in [0, 1]$

$$(a(x)u'(x))' = -p(x)$$

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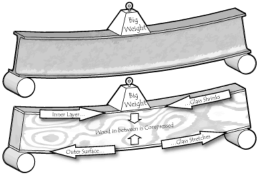

- Composite material characterized by the tensor $a(x)$
- Spatial variation: $a(x) = e^{f(x)}$, where $f(x)$ is a Gaussian process.

Material Failure

- Question: whether and where the material breaks.
- The conditional distribution of f conditional on the failure.

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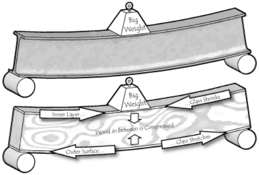
Material Failure – Dirichlet condition



- One dimensional problem: $(a(x)u'(x))' = -p(x)$
- Dirichlet condition: $u(0) = u(1) = 0$
- The solution:
  $u(x) = \int_0^x F(y)a^{-1}(y)dy - \frac{\int_0^1 F(y)a^{-1}(dy)dy}{\int_0^1 a^{-1}(dy)} \int_0^x a^{-1}(y)dy,$
  where $F(x) = \int_0^x p(y)dy$.

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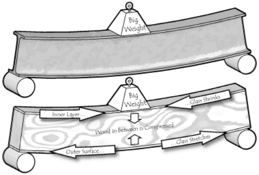

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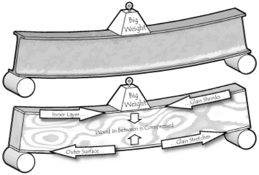

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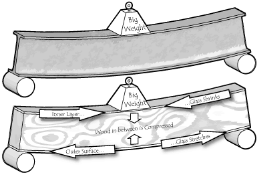
Material Failure – Dirichlet condition

- The strain

$$\begin{aligned} u'(x) &= a^{-1}(x)\left(F(x) - \frac{\int_0^1 F(y)a^{-1}(y)dy}{\int_0^1 a^{-1}(y)dy}\right) \\ &= a^{-1}(x)[F(x) - E_f(F(Y))] \end{aligned}$$

where $a^{-1}(x) = e^{f(x)}$.

The external force



- Delta external force:
  $p(x) = \delta_{x_*}(x), \quad F(x) = \int_0^x p(y)dy = I(x \geq x_*).$
- Continuous external force $p(x)$: $x_* = \arg\sup_{x \in T} |p(x)|$.

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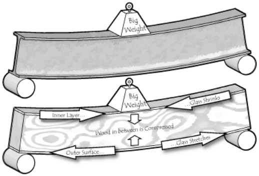

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Theorem: approximation of the Delta function (L. and Zhou 2011)

- Homogeneous, mean zero, and $C^3(T)$
- The covariance $C(t) = 1 - \frac{1}{2}t^2 + O(|t|^4)$.
- The external $F(x) = I(x \geq x_*)$, $\quad p(x) = \delta_{x_*}(x)$.

Theorem: approximation of the Delta function (L. and Zhou 2011)

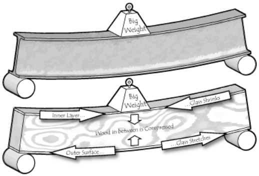
Let

$$Z \sim N(0,1), \quad H(x) = -\frac{x^2}{2} + \log P(Z \leq x), \quad \kappa = \sup H(x).$$

Let $r = \log b - \kappa$. Then, we have the approximation

$$P\Big(\sup_{x\in[0,1]} |u'(x)| > b\Big) \sim D \times P(Z > r).$$

Key components of the conditional distribution



- Questions about the conditional distribution
  - Where does the break occur or $\arg\sup u'(x) = ?$
  - Where does $f(x)$ attain it maximum?
  - At what level?

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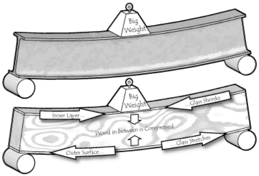

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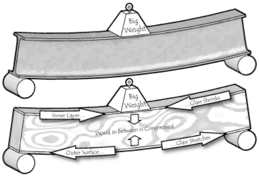

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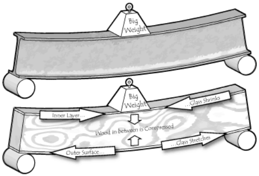

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Theorem: approximation for continuous force (L. and Zhou 2011)

The external force $p(x)$ is a continuously differentiable function. Then, we have the approximation

$$P\left(\sup_{x\in[0,1]}|u'(x)|>b\right) \sim P(|u'(0)|>b)+P(|u'(1)|>b)+P\left(\sup_{|x-x_*|<\varepsilon}|u'(x)|>0\right).$$

Exact asymptotic approximation for continuous body force

- Let $p(x_*)r^{-1}e^{r-\frac{1}{2}} = b$. Then,

$$P(\sup_{|x-x_*|<\varepsilon} |u'(x)| > 0) \sim \kappa_* \times r^{-1/2} \exp\{-r^2/2\}.$$

- Let $H_0 r_0{}^{-1/2} e^{r_0} = b$. Then,

$$P(|u'(0)| > b) = \kappa_0 \times r_0{}^{-1} e^{-r_0{}^2/2}$$

- Let $H_1 r_1{}^{-1/2} e^{r_1} = b$. Then,

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High dimensional case

- The partial differential equation: $x \in T$

$$\begin{cases} -\nabla \cdot \sigma(x) = p(x) \\ \sigma(x) = a(x)\nabla u(x) \end{cases}$$

with boundary condition that $u(\partial T) = 0$.

- The elasticity tensor $a(x) = e^{\xi(x)}$ and $Var(\xi(x)) = \sigma^2$.
- Conjecture:

$$\lim_{b\to\infty} \frac{\log P(\max_{x\in T} |u(x)| > b)}{(\log b)^2} = -\frac{1}{2\sigma^2}.$$

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The change-of-measure-based analysis

- Let P be the original measure.
- The change of measure Q

$$\frac{dQ}{dP} = \int_{t \in T} \frac{g_t(f(t))}{\varphi_t(f(t))} h(t) dt$$

where $\varphi_t(x)$ is the marginal density of $f(t)$, $h(t)$ is a density on T, and $g_t(x)$ is an alternative density.

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Simulation from the change of measure

- Simulate $\tau \in T$

$$\tau \sim h(t)$$

- Simulate $f(\tau)$ according to $g_t(x)$
- Simulate $\{f(t) : t \neq \gamma\}$ given $f(\tau)$ under P
- Simulation and computation

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The interpretations

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The choices of h and g_t

- Let $u = \log b$
- Random index

$$h(t) = \frac{P(f(t) > u - 1/u)}{\int_T P(f(t) > u - 1/u)dt} \propto P(f(t) > u - 1/u)$$

- The distribution $g_t(x) = \frac{I(x > u - 1/u)}{P(f(t) > u - 1/u)} \varphi_t(x)$

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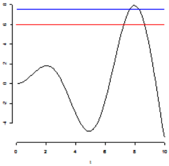
The choices of h and g_t

- The likelihood ratio:

$$\frac{dQ}{dP} = \frac{mes(A_{u-1/u})}{\int_T P(f(t) > u - 1/u)dt}$$

where $A_\gamma = \{t : f(t) > \gamma\}$.



Summary

- Stochastic partial differential equations and their physical interpretation.
- Gaussian processes are used to model the spatial variations.
- Asymptotic approximation.
- Simulation.